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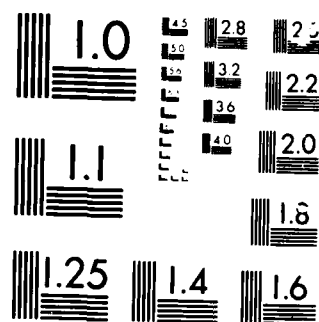
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This grant has sponsored the writing of 25 research papers. Recent titles this year include, "Linear estimation of boundary-value stochastic-processes in one-and-two-dimensions", "The reduction of perturbed Markov generators: an algorithm exposing the role of transient states", "Discrete-time Markovian jump linear quadratic optimal control", "An algebraic approach to the analysis and control of time scales", and "Tracking control of non-linear systems using sliding surfaces with applications to robot manipulators".

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Status Report on

ASYMPTOTIC METHODS FOR THE ANALYSIS,  
ESTIMATION, AND CONTROL OF STOCHASTIC  
DYNAMIC SYSTEMS

Grant AFOSR-82-0258

Covering the Period

November 11, 1984 to November 10, 1985

Prepared by

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SUMMARY

In this report we present a brief description of the research carried out by faculty, staff, and students in the M.I.T. Laboratory for Information and Decision Systems under Grant AFOSR-82-0258. The principal investigator for this research is Professor Alan S. Willsky, and the co-principal investigator is Prof. George C. Verghese. The time period covered in this status report is from November 11, 1984 to November 10, 1985.

The basic scope of this grant is to carry out fundamental research in the analysis, control, and estimation of complex systems, with particular emphasis on the use of methods of asymptotic analysis and multiple time scales to decompose complex problems into interconnections of simpler ones. During the time period covered by this report, significant progress has been made in several areas, leading to important results and to promising direction for further research.

The Specific topics covered in this report are:

- I. Analysis and Estimation for Finite-State and Hybrid Processes Possessing Time or Spatial Decompositions
- II. Analysis and Control of Singularly Perturbed and Weakly Coupled Linear Systems
- III. Analysis and Estimation for Singular Systems

A complete list of publications (completed and in preparation) supported by this grant is included at the end of this report.

I. Analysis and Estimation for Finite-State and Hybrid Processes Possessing Time of Spatial Decomposition

Our work in the past year in this portion of our project has focused on following through on the two major directions described in significant detail in our preceding status report and proposal: time scale decomposition of singularly-perturbed finite-state Markov processes (FSMP's); and modeling and estimation for spatially-distributed finite-state processes. Since detailed descriptions of these research projects are given in the previous status report and proposal, we present here abbreviated descriptions, together with discussions of the progress made in the last year.

Our work on time scale decompositions of FSMP's has dealt with a model of the form

$$\dot{x}(t) = A(\epsilon)x(t) \quad (1.1)$$

where  $A(\epsilon)$  is an infinitesimally stochastic matrix,  $\epsilon$  is a small parameter, and  $x(t)$  is the probability vector of the FSMP. In our early work in this area [1], [2], [4] we had developed a general procedure for constructing a multiple time scale decomposition of (1.1) and for obtaining aggregated models at successively slower time scales. Specifically, it is straightforward to check that  $A(0)$  captures the fast time scale behavior (i.e. transitions in the FSMP that occur in order 1 time rather than  $1/\epsilon$  or longer). Then, if we let  $P(\epsilon)$  denote the projection onto the eigenspace of all eigenvalues of  $A(\epsilon)$  that are  $O(\epsilon)$ , we can define another matrix

$$A_1(\epsilon) = \frac{P(\epsilon)A(\epsilon)}{\epsilon} \quad (1.2)$$

so that we can repeat the procedure at the next time scale: i.e.  $A_1(0)$  captures transition behavior of the original process at the time scale  $1/\epsilon$ .

and

$$A_2(\epsilon) = \frac{P_1(\epsilon)A_1(\epsilon)}{\epsilon} \quad (1.3)$$

contains all the information on slower behavior, where  $P_1(\epsilon)$  is the corresponding projection for  $A_1(\epsilon)$ .

There are several major limitations to this previous work. Specifically, the calculation of the full  $\epsilon$ -dependent projection  $P(\epsilon)$  is a highly nontrivial computation, without a simple probabilistic interpretation. Furthermore, while the methods of [1], [1], [4] provide a procedure for aggregation, this aggregation is accomplished after the fact -- i.e. the computations (2.2), (2.3), etc. must be carried out on the full process and only at the end can one perform aggregation.

This is in marked contrast to results of others which apply only to a rather restrictive subclass of models as in (1.1). Specifically, under the condition known as "nearly complete decomposability" on  $A(\epsilon)$  -- in which the states at each time scale can be grouped into ergodic classes so that there are no transient states and transitions between classes occur only at slower time scales -- one can proceed as follows. Note that  $P(0)$  is nothing more than the ergodic projection of  $A(0)$ . If  $A(\epsilon)$  is nearly completely decomposable, then (2.2) can be replaced by

$$\hat{A}_1(\epsilon) = \frac{P(0)A(\epsilon)}{\epsilon} \quad (1.4)$$

In fact, because of the simple interpretation of  $P(0)$  we can go one step farther. Specifically, since  $P(0)$  is the ergodic projection of an FSMP without transient states, it can be written as

$$P(0) = UV \quad (1.5)$$

where  $V$  is a matrix of 1's and 0's, where the 1's in a particular row correspond to the states that form a single ergodic class in  $A(0)$  and where the corresponding column of  $U$  is the vector of ergodic probabilities assuming the process starts in this class. From this one can deduce that  $VU = I$  and that the slow time behavior captured by  $\hat{A}_1(\epsilon)$  is also captured by

$$F_1(\epsilon) = \frac{VA(\epsilon)U}{\epsilon} \quad (1.6)$$

which is a Markov generator on an aggregated state space with one aggregated state per ergodic class of  $A(0)$  and with transition rates between aggregates representing average transition rates from states in one ergodic class to states in another, where the averaging is done using the ergodic probabilities of  $A(0)$  (i.e. elements of  $U$ ).

As pointed out in the previous status report, the simplified procedure just described can break down if there are transient states, at any time scale, as the averaging implied by  $P(0)$  will miss potentially important couplings of aggregated states through transient states. It is this phenomenon that requires the retention of at least some  $\epsilon$ -dependent terms in  $P(\epsilon)$ . In our recently completed paper [16] (see also [12], [25]) we have succeeded in solving the same general problem as in [1], [2], [4] but with a far simpler procedure that maintains the advantages of the procedure developed in the nearly completely decomposable case: the computations are straightforward, with clear probabilistic interpretations, and at each stage of the procedure we work on increasingly aggregated versions of the original process. The proof of this result involves rather delicate arguments and careful accounting for the dominant transition paths between states (and hence there is a graph-theoretic flavor to the result). The end result of this



analysis is that in the general case, the slow time scale behavior of (1.1) is captured by an aggregated Markov generator

$$G_1(\epsilon) = \frac{V(\epsilon)A(\epsilon)U}{\epsilon} \quad (1.7)$$

where  $U$  is as before, and where the "ergodic class membership matrix"  $V(\epsilon)$  is now not simply a 0-1 matrix but is in fact  $\epsilon$ -dependent. This  $\epsilon$ -dependence captures the fact that some transient states of  $A(0)$  may in fact have  $\epsilon$ -dependent transitions into more than one ergodic class of  $A(0)$ , and therefore the "membership" of this class must be split accordingly. As discussed in [16], the elements of  $V(\epsilon)$  can be interpreted and calculated as trapping probabilities of simplified FSMP's and in fact all that is needed is to make sure that the leading order terms of  $V(\epsilon)$  match those of the corresponding trapping probabilities and that the columns of  $V(\epsilon)$  all add up to 1 (corresponding to the "total membership" of each state in the original chain). The result is a straightforward computational procedure described in detail in [16].

The result just described represents a significant breakthrough for several reasons. First, we now have a computationally feasible method for multiple time scale decomposition of FSMP's, and this opens the door for the consideration of the application of this theory to a variety of problems, ranging from reliability analysis to analysis of networks of queues. Also, the theoretical machinery we have developed should allow us to make significant extensions of these results -- to discrete-time chains, perturbed semi-Markov processes, and to interconnected systems described as in (1.1) but in which  $A(\epsilon)$  is not a Markov generator and  $x(t)$  is the state of a system rather than a probability vector. A number of these research directions are

elaborated upon in significant detail in [25].

The other portion of our research during the past year focused on the development of a modeling methodology and estimation strategies for extremely complex event-driven systems. As discussed in our preceding status report, the modeling of cardiac activity and electrocardiogram (ECG) analysis provided us with an excellent context for this study, since distributed models of the heart and the ECG analysis problem capture many of the key features found in numerous applications involving complex signal analysis or the monitoring of distributed systems. Specifically, our work has focused on the modeling of coordination, or more specifically timing and control, in interconnected systems and on the development of distributed estimation structures for such systems. The latter problem has as a major component the design of coordination strategies for the processors charged with monitoring individual subsystems. A second major problem we have addressed in our investigation is the development of meaningful measures of performance for such event-oriented estimation problems.

During the past year we have completed a major research project in this area. This research is described in the Ph.D. thesis [10]. Also, we have completed one paper [15] detailing our approach to constructing distributed models of cardiac activity, and a second paper [18] describing our methodology for the design and performance evaluation of distributed estimators for such models of interconnected systems is in preparation. A variety of directions for further research have been identified and documented [10].

## II. Analysis and Control of Singularly Perturbed and Weakly Coupled Linear Systems

A major part of our research has been devoted to developing an algebraic approach to the study of time-scale structure in perturbed, linear, time-invariant systems of the form

$$\dot{x}(t) = A(\epsilon)x(t) + B(\epsilon)u(t) \quad (2.1)$$

where  $\dot{x}$  is the N-dimensional state vector,  $u$  the m-dimensional vector of control inputs, and  $\epsilon$  a (small, positive) perturbation parameter; the entries of  $A(\epsilon)$  and  $B(\epsilon)$  (as well as of other functions of  $\epsilon$  that are introduced later) are taken from the ring of functions of  $\epsilon$  that are analytic at 0. Our accomplishments so far in this direction are contained in the recently completed thesis [9]. A summary of those results of the thesis obtained during the last year is presented first.

A second focus for our work in this past year on systems of the form (2.1) has been the study of their orders of controllability. Considerable progress has been made on questions we raised earlier in this regard, and this is summarized in the remainder of the section.

### 1. Time-Scale Structure

Consider first the undriven form of (2.1), namely

$$\dot{x}(t) = A(\epsilon)x(t) \quad (2.2)$$

Assume  $A(\epsilon)$  to be Hurwitz for  $\epsilon$  in  $(0, \epsilon_0]$ ; the extension to the more general case where  $A(\epsilon)$  is semistable is described in [9]. If  $A(0)$  is singular, the system (2.2) is termed singularly perturbed, because some eigenvalues that are nonzero for  $\epsilon \neq 0$  become 0 at  $\epsilon = 0$ , causing the matrix  $A(\epsilon)$  to become

singular at  $\epsilon = 0$ . We consider only this singularly perturbed case in this section.

There has been much interest in the question of when (2.2) has well-defined time-scale structure, as defined by Coderch et al. in [1]. The objective is to expose and exploit this structure by obtaining a time-scale decomposition of the system. It was shown in [4] that (2.2) has well-defined time-scale structure if and only if  $A(\epsilon)$  satisfies a so-called multiple semistability or MSST condition.

Our algebraic approach to this problem begins by performing a unimodular similarity transformation of (2.2):

$$x(t) = P(\epsilon)y(t), \quad \det[P(0)] = \text{nonzero constant} \quad (2.3)$$

The above condition on  $P(\epsilon)$  causes it to be a well behaved transformation at  $\epsilon = 0$ , because its inverse is well defined there.  $P(\epsilon)$  is chosen such that the transformed system takes the so-called explicit form

$$\dot{y}(t) = D(\epsilon)\bar{A}(\epsilon)y(t) \quad (2.4)$$

where  $\bar{A}(\epsilon)$  is also unimodular, i.e.  $\det[\bar{A}(0)] = \text{nonzero constant}$ , and where

$$D(\epsilon) = \text{diag} \{ \epsilon^{k_1} I_{i_1}, \dots, \epsilon^{k_n} I_{i_n} \}, \quad k_1 < \dots < k_n \quad (2.5)$$

The diagonal elements of  $D(\epsilon)$  are the invariant factors of  $A(\epsilon)$ . The existence of a Smith decomposition for any  $A(\epsilon)$  that has entries analytic at 0 is what guarantees the existence of such a transformation.

What underlies many of our results now is the idea that, since  $\bar{A}(\epsilon)$  is nonsingular at  $\epsilon = 0$ , any time-scale structure of the system (2.4) must be explicitly displayed by the matrix  $D(\epsilon)$ . (This would be trivially true if we had  $\bar{A}(\epsilon) = I$ .) This time-scale structure should in turn reflect that of the underlying system (2.2), because the transformation matrix  $P(\epsilon)$  is unimodular.

Carrying the same idea one step further, one might believe that significant error will not be incurred if  $\bar{A}(\epsilon)$  in (2.4) is replaced by  $\bar{A}(0)$ , yielding what we call the reduced explicit form of (2.2):

$$\dot{z}(t) = D(\epsilon)\bar{A}(0)z(t) \quad (2.6)$$

The results below show to what extent these intuitive expectations are fulfilled.

Let  $\bar{A}(0)$  in (2.6), henceforth represented simply by  $\bar{A}$ , be partitioned conformably with  $D(\epsilon)$ :

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & \cdot & \cdots & A_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ A_{n1} & \cdot & \cdots & A_{nn} \end{bmatrix} \quad (2.7)$$

Denote the Schur complement of  $A_{11}$  in  $\bar{A}$  by  $C_{22}$ , and let its leading block entry be denoted by  $\tilde{A}_{22}$  ( $= A_{22} - A_{21}A_{11}^{-1}A_{12}$ ). Let  $C_{33}$  then denote the Schur complement of  $\tilde{A}_{22}$  in  $C_{22}$ , and denote its leading block matrix entry by  $\tilde{A}_{33}$ . Construct  $\tilde{A}_{ii}$  for  $i > 3$  similarly. It is then shown in [9] that  $A(\epsilon)$  in (2.2) satisfies the MSST condition of [1] if and only if all the  $\tilde{A}_{ii}$  for  $i \geq 1$  are Hurwitz (where  $\tilde{A}_{11} = A_{11}$ ). Furthermore, the multiple semi-simple null structure or MSSNS condition of [1] is shown to be equivalent to nonsingularity of the  $\tilde{A}_{ii}$ . Under the MSST condition, we have shown how to construct a constant matrix  $T$  such that

$$\lim_{\epsilon \rightarrow 0} \sup_{t \geq 0} \|\exp\{A(\epsilon)t\} - T \exp\{\hat{A}(\epsilon)t\} T^{-1}\| = 0 \quad (2.8)$$

where

$$\hat{A}(\epsilon) = \text{diag} \{ \epsilon^{k_1} \tilde{A}_{11}, \dots, \epsilon^{k_n} \tilde{A}_{nn} \} \quad (2.9)$$

This leads directly to an explicit time-scale decomposition of the system (2.2).

Last year's proposal noted that we seemed to have ready the seeds of an extension of time-scale decomposition ideas to systems that do not possess well-defined time scales in the sense of [1]. An important accomplishment of our research in this past year has been to obtain results along these lines, fulfilling the expectations listed last year. To see the issues involved, consider the following example.

Example 1 Suppose

$$A(\epsilon) = \begin{bmatrix} -\epsilon - \epsilon & 1 \\ -1 & -\epsilon - \epsilon^2 \end{bmatrix} \quad (2.10)$$

Here  $A(\epsilon)$  is already in explicit form, and

$$A_{11} = A(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (2.11)$$

is nonsingular but (unlike  $A(\epsilon)$  for small positive  $\epsilon$ ) is not Hurwitz. In other words,  $A(\epsilon)$  satisfies the MSSNS but not the MSST condition. The associated system (2.2) therefore does not have well-defined time scales in the sense of [1], and there is no constant matrix  $A_0$  such that

$$\lim_{\epsilon \rightarrow 0} \sup_{t \geq 0} \| \exp\{A(\epsilon)t\} - T \exp\{A_0 t\} T^{-1} \| = 0 \quad (2.12)$$

The problem is that the system has a damping of order higher than its natural oscillation frequency, so that its solutions are of the form

$$x_i(t) = e^{-\epsilon t - \epsilon^2 t} \sin(t + \theta) \quad (2.13)$$

If, however, we used the  $\epsilon$ -dependent matrix

$$A_0(\epsilon) = \begin{bmatrix} -\epsilon & 1 \\ 1 & -\epsilon \end{bmatrix} \quad (2.14)$$

instead of  $A_0$  in (2.12), we would find that the equality now does hold.

The above example suggests that if we keep some  $\epsilon$ -dependent terms in our system we may obtain valid time-scale decompositions even if the system satisfies only MSSNS and not MSST. Following up on this idea, we shall say that the system (2.2) has extended well-defined time scales if there exist unimodular matrices  $A_i(\epsilon)$ ,  $i = 1$  to  $n$ , and a constant invertible matrix  $T$  such that (2.9) holds, with  $\hat{A}(\epsilon)$  now defined by

$$\hat{A}(\epsilon) = \text{diag} \{ \epsilon^{k_1} A_1(\epsilon), \dots, \epsilon^{k_n} A_n(\epsilon) \} \quad (2.15)$$

instead of by (2.9). Note that (2.8) and (2.15) then lead directly to the extended time-scale decomposition. The thesis [9] describes how, under the assumption that  $A(\epsilon)$  satisfies MSSNS (and with the standing assumption that it is Hurwitz for a range of  $\epsilon > 0$ ), one can always obtain such an extended time-scale decomposition.

Our work over the last year has also rounded out several of our earlier results on  $\epsilon$ -dependent amplitude scaling of systems of the form (2.2) in order to achieve MSSNS in systems that do not already satisfy this condition. The importance of MSSNS is clear from the above results on extended time-scale

decomposition. Furthermore, we have earlier shown, see [12], that under MSSNS one can approximate the eigenvalues of  $A(\epsilon)$  via those of  $\epsilon^{k1} \tilde{A}_{11}, \dots, \epsilon^{kn} \tilde{A}_{nn}$ , where the matrices  $\tilde{A}_{ii}$  are as defined above. The thesis [9] describes a systematic scaling procedure, along with certain conditions sufficient to ensure that the procedure results in a MSSNS system. Though the sufficient conditions are rather strong (reflecting what we believe to be the intrinsic difficulty of the problem), we have found that they are applicable to the sorts of high-gain feedback problems considered by Sannuti and Wason (IEEE Trans. Auto. Control, AC-30, 7, 633-644, July 1985), for example.

## 2. Orders of Controllability

Major progress has been made during the last year on the problem of defining and understanding the structure and significance of "orders of controllability" in systems of the form (2.1). We believe we have now found a sound, satisfying and self-consistent definition of this notion, and this will be outlined here. Our development has been carried out in the context of the discrete-time system

$$x[k+1] = A(\epsilon)x[k] + B(\epsilon)u[k] \quad (2.16)$$

but a similar development can be carried out for the continuous-time model (2.1). (Difference between the considerations for discrete- and continuous-time systems may be expected to appear when we examine the interaction between orders of controllability and feedback control of time-scales).

We assume that the system is controllable for a range of  $\epsilon > 0$ . The set of states controllable (or, more correctly, reachable) from the origin in  $N$  steps, starting at time  $k = 0$ , is given by



$$x[N] = C_N(\epsilon)U[N] \quad (2.17a)$$

where

$$C_N(\epsilon) = [B(\epsilon), A(\epsilon)B(\epsilon), \dots, A^{N-1}(\epsilon)B(\epsilon)] \quad (2.17b)$$

and

$$U[N] = \begin{bmatrix} u[N-1] \\ u[N-2] \\ \vdots \\ u[0] \end{bmatrix} \quad (2.17c)$$

We now say a target state  $x[N]=x(\epsilon)$ , with entries analytic at 0, is  $\epsilon^j$ -controllable if there is a control sequence  $U[N] = U(\epsilon)$ , also analytic at 0, such that

$$\epsilon^j x(\epsilon) = C_N(\epsilon)U(\epsilon) \quad (2.18)$$

The following examples will help to make this definition more concrete.

Example 2 Suppose

$$A(\epsilon) = \begin{bmatrix} 1 & 1 \\ \epsilon & 2 \end{bmatrix} \quad B(\epsilon) = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$$

so that

$$C_N(\epsilon) = \begin{bmatrix} 1 & 1+\epsilon \\ \epsilon & 3\epsilon \end{bmatrix}$$

It is then evident that the target state  $[1 \ 0]^T$  is  $\epsilon^0$ -controllable, while the target state  $[1 \ 1]^T$  is only  $\epsilon^1$ -controllable. (In the limit of  $\epsilon = 0$ , of

course, the first target state remains controllable while the second target state is no longer controllable.)

Example 3 If, in  $A(\epsilon)$  of the above example, we change the  $\epsilon$  to  $-\epsilon$ , then

$$C_N(\epsilon) = \begin{bmatrix} 1 & 1+\epsilon \\ \epsilon & \epsilon \end{bmatrix}$$

We now find that, while  $[1 \ 0]^T$  is still  $\epsilon^0$ -controllable, the target  $[1 \ 1]^T$  is only  $\epsilon^2$ -controllable.

The set of  $\epsilon^j$ -controllable states is denoted by  $X^j$ , and constitutes the  $\epsilon^j$ -controllable submodule; it is  $A(\epsilon)$ -invariant. The following inclusion property is then evident from the definition:

$$X^0 \subset X^1 \subset \dots \quad (2.19)$$

We have shown that the structure of these submodules is determined by the invariant factor structure of  $C_N(\epsilon)$ , i.e. by the Smith form of  $C_N(\epsilon)$ . As with our time-scale studies, therefore, we are very naturally led to an algebraic characterization and study of the problem. To see the reason for the role of the Smith form, let

$$C_N(\epsilon) = R(\epsilon)M(\epsilon)Q(\epsilon) \quad (2.20a)$$

be the Smith decomposition of  $C_N(\epsilon)$ , where  $R(\epsilon)$  and  $Q(\epsilon)$  are unimodular matrices, and

$$M(\epsilon) = [\bar{M}(\epsilon) \ 0], \quad (2.20b)$$

$$\bar{M}(\epsilon) = \text{diag} \{ \epsilon^{s_1} I_{m_1}, \dots, \epsilon^{s_p} I_{m_p} \}, \quad s_1 < \dots < s_p \quad (2.20c)$$

The structure of the  $\epsilon^i$ -controllable submodules can then be read off directly from the matrix  $R(\epsilon)\bar{M}(\epsilon)$ . Rather than describing this here in general

notation, we simply illustrate the results on the two examples above.

Example 2a For Example 2 we have

$$C_N(\epsilon) = I \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} 1 & 1+\epsilon \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} Q(\epsilon)$$

as the Smith decomposition (2.20a). We then have an  $\epsilon^0$ -controllable submodule of the form  $[x_1(\epsilon) \ \epsilon x_2(\epsilon)]^T$ , where  $x_1(\epsilon)$  and  $x_2(\epsilon)$  are analytic at 0 but otherwise arbitrary, and an  $\epsilon^1$ -controllable submodule of the form  $[x_1(\epsilon) \ x_2(\epsilon)]^T$ .

Example 3a For Example 3 we have

$$C_N(\epsilon) = \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2 \end{bmatrix} \begin{bmatrix} 1 & 1+\epsilon \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \epsilon & \epsilon^2 \end{bmatrix} Q(\epsilon)$$

There is now an  $\epsilon^0$ -controllable submodule of the form  $[x_1(\epsilon) \ \epsilon x_1(\epsilon) + \epsilon^2 x_2(\epsilon)]^T$ , an  $\epsilon^1$ -controllable submodule of the same form, and an  $\epsilon^2$ -controllable submodule of the form  $[x_1(\epsilon) \ x_2(\epsilon)]^T$ .

The Smith decomposition also allows us to transform the original system (2.16) into a standard form of the type that was conjectured in our proposal last year (written below for the case where  $s_i = i-1$ , to keep the notation simple)

$$R^{-1}(\epsilon)A(\epsilon)R(\epsilon) = \begin{bmatrix} A_{11}(\epsilon) & A_{12}(\epsilon) & \dots & A_{1p}(\epsilon) \\ \epsilon A_{21}(\epsilon) & A_{22}(\epsilon) & \dots & A_{2p}(\epsilon) \\ \epsilon^2 A_{31}(\epsilon) & \epsilon A_{32}(\epsilon) & \dots & A_{3p}(\epsilon) \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}, \quad R^{-1}(\epsilon)B(\epsilon) = \begin{bmatrix} B_1(\epsilon) \\ \epsilon B_2(\epsilon) \\ \epsilon^2 B_3(\epsilon) \\ \vdots \end{bmatrix} \quad (2.21)$$

With this standard form in hand, we are in a position to more simply discuss issues such as eigenvalue placement by state feedback. The eigenvalues of  $A(\epsilon)$  are clustered around those of the  $A_{ii}(\epsilon)$  in (2.21) above. It turns out now, as might be expected, that the eigenvalues of  $A_{ii}(\epsilon)$  can be shifted by  $O(1)$  using state feedback if and only if we use feedback gains that are  $O(\epsilon^{1-i})$ . We have obtained an algorithm to pick, in a way that is directed by the system structure, a feedback gain that effects a shift of all the eigenvalues to desired positions.

We believe the road is now open for several useful and interesting results to be obtained, aimed at structuring the design and implementation of feedback control in accordance with (not only time-scale structure but also) the orders of controllability of the system. Directions for further research in this vein are discussed in the accompanying proposal.

### III. Singular Systems

During the past year we have continued our research on the class of discrete-time, boundary-value singular models of the form

$$Ex(k+1) = Ax(k) + Bu(k) \quad (3.1)$$

$$v = V_0x(0) + V_Nx(N) \quad (3.2)$$

In the year preceding this past year we had made significant progress in analyzing the estimation problem for such systems, and this analysis raised a number of open questions that related directly to the fundamental properties -- well-posedness, stability, minimality, etc. -- of models of this type. For this reason we focused most of our efforts during the past year on the development of a complete system theory for models as in (3.1), (3.2), and in this area we have made substantial progress. Our results in this area are outlined in the S.M. thesis proposal [24] and will be described more completely in the completed thesis and in a paper to be written on this subject.

One of the basic results we have obtained and used heavily in our work is the following. Suppose that  $(zE-A)$  is a regular pencil (i.e. its determinant is not identically zero). Then (3.1) (3.2) is well posed if and only if  $(V_0E^N + V_NA^N)$  is invertible. In this case, we can always put the system in standard form, so that

$$V_0E^N + V_NA^N = I \quad (3.3)$$

$$\alpha E + \beta A = I \quad (3.4)$$

for some pair of real numbers  $\alpha$  and  $\beta$  (note that (3.4) implies that  $EA = AE$ ).

This result by itself answers an important question posed in our previous proposal, and, more importantly, it has opened the way for the development of

a complete theory for this class of systems. In particular, we have developed results for controllability, observability, and minimality for these systems. While previous efforts have produced results on these concepts, they are significantly different from our results for two reasons. The first of these is that our use of standard form allows us to obtain far more compact and easily understood conditions using a generalized Cayley-Hamilton theorem for regular pencils in standard form. The second is that our investigation of discrete-time singular models is unique thanks to our inclusion of the general boundary condition (3.2) (which seems to be the natural choice given the intrinsic noncausality of (3.1)).

An important property of the class of systems described by models as in (3.1), (3.2) is that they require two notions each for controllability and observability. In particular, we can define an inward boundary process obtained by using the original boundary condition (3.2) and the inputs near the boundary, i.e.  $u(k)$ ,  $k \in [0, k_0) \cup (k_1, N-1]$  to propagate the boundary conditions in to corresponding constraints on  $x(k_0)$  and  $x(k_1)$ . Similarly, we can define a map propagating outward from the center of the interval. Using these constructions, we have obtained results on controllability for each of these mappings and for corresponding notions of observability. Our work here has paralleled that of Krener in his study of nonsingular continuous-time boundary-value models (i.e.  $\dot{x} = Ax + Bu$ , with boundary conditions analogous to (3.2)), although the possible singularity of both  $E$  and  $A$  leads to several important differences in our theory. An interesting and perhaps surprising part of Krener and our theories is the more complicated nature of minimality. Specifically, minimality does not require both controllability maps to be onto

nor both observability maps to be one-to-one, and furthermore alternate minimal realizations may not be related by a similarity transformation.

In addition to the results cited above, we have obtained further results in the case of stationary processes -- i.e. models for which the weighting pattern kernel from  $u$  in (3.1), (3.2) to

$$y = Cx \quad (3.5)$$

is shift-invariant. Note that this isn't generally the case even with  $A$ ,  $B$ ,  $C$  constant and it in fact requires the commutativity of  $E$ ,  $A$  with  $V_0$  and  $V_N$ .

Our development of a system theory for boundary-value singular systems has also led to new results in two additional areas. First, we have derived and have begun to analyze a boundary-value Lyapunov equation for stationary models with  $u(k)$  a white sequence with covariance  $Q$  and  $v$  an independent random variable with covariance  $\Pi$ . This equation differs considerably from any that have appeared in the control-oriented descriptor literature.

Once one has a Lyapunov equation, it is natural to ask questions about stability of models as in (3.1), (3.2). However, stability refers to an asymptotic property of a recursion, and this raises two questions: what are "recursive" solutions of (3.1), (3.2) and how do we deal with asymptotic properties when the interval of interest,  $[0, N]$ , is bounded. We have made significant progress on both of these questions. Specifically, in [17], [19] we discuss a "two-filter" solution to (3.1), (3.2) obtained by decoupling (3.1) into a causal and anticausal part. These two parts can always be made stable (in the usual discrete-time linear system sense) if  $|zE-A|$  has no roots on the unit circle. While this is interesting (and is related to previous stability analyses of descriptor systems) it is not completely satisfying.

used in previous work (on standard, non-singular systems) to obtain recursive implementations of the optimal smoother. Our results will be described in detail in [19]. Beyond our results to date, there is much left to be done in improving our understanding of the structure of the estimation problem for these systems and in particular in analyzing the singular Riccati equations that arise in this procedure and their relationship to the somewhat different equations that have appeared in other, control-oriented studies of singular models. In addition, we have continued our work on estimation for 2-D singular models. In particular [17] contains a description of our work on a class of such models with sufficient spatial symmetry to allow us to obtain an extremely efficient solution to the smoothing problem. This is obtained by decoupling the dynamics in one spatial variable through the use of a discrete Fourier transform and then using the Hamiltonian diagonalization procedure to obtain an efficient method for solving the dynamics in the other variable. Much still remains to be done in this area in considering more general models and in obtaining more efficient methods to account for the effect of boundary conditions.



### PUBLICATIONS

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1. M. Coderch, A.S. Willsky, S.S. Sastry, and D.A. Castanon, "Hierarchical Aggregation of Linear Systems with Multiple Time Scales," IEEE Trans. on Automatic Control, Vol. AC-28, No. 11, Nov. 1983, pp. 1017-1030.
2. M. Coderch, A.S. Willsky, S.S. Sastry, and D.A. Castanon, "Hierarchical Aggregation of Singularly Perturbed Finite State Markov Processes," Stochastics, Vol. 8, 1983, pp. 259-289.
3. D.A. Castanon, M. Coderch, A.S. Willsky, "Hierarchical Aggregation of Diffusion Processes with Multiple Equilibrium Points," Proc. of the 1982 IEEE Conf. on Dec. and Control; extended version in preparation.
4. M. Coderch, "Multiple Time Scale Approach to Hierarchical Aggregation of Linear Systems and Finite State Markov Processes," M.I.T. Lab. for Inf. and Dec. Sys. Rept. LIDS-TH-1221, August 1982.
5. S.S. Sastry, "The Effects of Small Noise on Implicitly Defined Non-linear Dynamical Systems," IEEE Trans. Circuits and Systems, Vol. CAS-30, No. 9, Sept. 1983, pp. 651-663.
6. J.J. Slotine and S.S. Sastry, "Tracking Control of Non-linear Systems Using Sliding Surfaces with Application to Robot Manipulators," International Journal of Control, Vol. 38, No. 2, Aug. 1983, pp. 465-492.
7. X.-C. Lou, G.C. Verghese, A.S. Willsky, and M. Vidyasagar, "An Algebraic Approach to the Analysis and Control of Time Scales," Proceedings of the 1984 American Control Conference, San Diego, Calif. June 1984.
8. P.C. Doerschuk, R.R. Tenney, and A.S. Willsky, "Estimation-Based Approaches to Rhythm Analysis in Electrocardiograms," Proc. of C.N.R.S. Meeting on Abrupt Changes in Signals and Systems, Paris, France, March 1984.
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22. M.B. Adams, B.C. Levy, and A.S. Willsky, "Estimation for Boundary-Value Processes in Two Dimensions," in preparation; to be submitted to IEEE Trans. Acoustics, Speech, and Sig. Proc.
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